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LETTER TO THE EDITOR

Dodd–Bullough equation, boundary condition and nonlocal conservation laws

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Abstract. We have reconsidered the integrability of the Dodd–Bullough equation in the presence of the boundary condition. It is shown that the boundary condition itself can be deduced from the Lax equation. The forms of the infinite number of nonlocal conservation laws are shown to change due to the presence of the boundary condition.

Integrable systems have been studied for the last few decades mainly from the viewpoint of Hamiltonian structure, symmetry, and quantization [1]. But it should be mentioned that the majority of such studies deal mainly with the problem on the full real line. The importance of the boundary condition has been realized only very recently. The first major step was taken by Sklyanin [2] who showed how effectively one could introduce a boundary condition and still retain integrability. He also gave an elegant prescription for the quantization of the system with the use of the corresponding classical and quantum γ-matrix. Of late, another approach has been put forward by Habibullin and others [3] to study the compatibility of a boundary condition through the use of Lie–Backlund symmetry [4] or otherwise [5]. In the present work we show how the required boundary condition itself can be deduced from a modified version of the Lax pair and how it effects the form of nonlocal conservation laws for the well-studied system Dodd–Bullough equation [6].

The equation under consideration can be written as

$$\partial_+\partial_-\phi = 2\mathrm{i}\pi\beta[\exp(\mathrm{i}\beta\phi) - 2\exp(-2\mathrm{i}\beta\phi)] \tag{1}$$

for which the Lax pair is written as

$$\begin{aligned} A_+(\lambda, x) &= -\frac{\beta}{2\mathrm{i}}\partial_+\phi H_1 - \sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda F_1 - 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)F_2 \\ A_-(\lambda, x) &= \frac{\beta}{2\mathrm{i}}\partial_-\phi H_1 + \sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}E_1 + 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)E_2 \end{aligned} \tag{2}$$

where ∂_+ and ∂_- are derivatives with respect to the light cone coordinates and $(H_1, F_1, E_1, F_2, E_2)$ the generators of the algebra. The generators have the matrix form written as

$$H_1 = e_{11} - e_{33} \quad F_1 = e_{21} - e_{32} \quad E_2 = e_{31} \quad E_1 = e_{12} - e_{23} \quad F_2 = e_{13} \tag{3}$$

where e_{pq} is a 3×3 matrix with one at the intersection of the pth row and qth column and zero elsewhere. The Lax equations leading to equation (1) are

$$(\partial_\pm - A_\pm)\psi(x_+, x_-) = 0. \tag{4}$$

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The compatibility condition of these reduces to

$$F = \partial_+ A_- - \partial_- A_+ + [A_-, A_+] = 0.$$

It is interesting to observe that we can have another set of Lax pairs for the same equation (1), which can be written as,

$$(\partial_\pm - \bar{A}_\mp)\tilde{\psi} = 0 \tag{5}$$

where the consistency reads

$$\tilde{F} = \partial_+ \bar{A}_+ - \partial_- \bar{A}_- + [\bar{A}_+, \bar{A}_-] = 0 \tag{6}$$

with

$$\begin{aligned} \bar{A}_+(\lambda, x) &= -\frac{\beta}{2\mathrm{i}}\partial_-\phi H_1 - \sqrt{2\pi}\beta \exp(\mathrm{i}/2\beta\phi)\lambda F_1 - 2\sqrt{\pi}\beta \exp(-\mathrm{i}\beta\phi) F_2 \\ \bar{A}_-(\lambda, x) &= \frac{\beta}{2\mathrm{i}}\partial_-\phi H_1 + \sqrt{2\pi}\beta \exp(\mathrm{i}/2\beta\phi)\lambda^{-1} E_1 + 2\sqrt{\pi}\beta \exp(-\mathrm{i}\beta\phi) E_2. \end{aligned} \tag{7}$$

To introduce the boundary conditions compatible with the complete integrability of the system we modify the gauge fields in the following manner:

$$\begin{aligned} A_+ &= \theta(x_1) A_+(\lambda, x) \\ \bar{A}_+ &= \theta(x_1) \bar{A}_+(\lambda, x) \\ A_- &= \theta(x_1) A_-(\lambda, x) - \delta(x_1)[f_1 H_1 + \lambda^{-1} f_2 E_1 + f_3 E_2] \\ \bar{A}_- &= \theta(x_1) \bar{A}_-(\lambda, x) - \delta(x_1)[f_1 H_1 + \lambda^{-1} f_2 E_1 + f_3 E_2] \end{aligned} \tag{8}$$

where $\theta(x_1)$ is the step function such that $\theta(x_1) = 1$, for $x_1 > 0$ and equal to zero for $x_1 < 0$. Here x_1 is the point of the real line where we want to impose the boundary condition, and f_1, f_2, f_3 are the three unknown functions to be determined and used in the boundary condition. We now impose the condition that

$$(F - \tilde{F})|_{x_1=0} = 0. \tag{9}$$

Collecting the coefficient of E_2 in equation (9) we get

$$(\partial_+ + \partial_-) f_3 = \frac{\beta}{2\mathrm{i}} f_3 (\partial_+ + \partial_-)\phi \tag{10}$$

whereas the coefficient of F_1 yields

$$f_1 = 0. \tag{11}$$

On the other hand, the coefficient of E_1 leads to

$$(\partial_+ + \partial_-) f_2 = \frac{\beta}{2\mathrm{i}} f_2 (\partial_+ + \partial_-)\phi. \tag{12}$$

So equations (10)–(12) complete the determination of the unknown functions f_1, f_2, f_3. It is then interesting to observe that the coefficient of F_2 gives us the boundary condition

$$(\partial_+ - \partial_-)\phi = 2\sqrt{\pi}(\sqrt{2}\mathrm{e}^{-\frac{3\mathrm{i}\beta}{2}\phi} - \mathrm{e}^{\frac{\mathrm{i}3\beta}{4}\phi}) \tag{13}$$

at the point $x = x_1$.

Let us now write the Lax equations

$$\begin{aligned} &(\partial_+ - A_+)\psi = 0 \qquad (\partial_- - A_-)\psi = 0 \\ &(\partial_+ - \bar{A}_-)\tilde{\psi} = 0 \qquad (\partial_- - \bar{A}_+)\tilde{\psi} = 0 \end{aligned} \tag{14}$$

in full form and convert them to the Riccati equation in the variables

$$V_1 = \psi_2/\psi_1 \qquad V_2 = \psi_3/\psi_1 \qquad \text{and} \quad u = \log \psi_1 \tag{15}$$

and similarly for the barred variables, whence we get

$$\partial_+ V_1 - \theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_+\phi V_1 + 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)V_1V_2 - \sqrt{2\pi}\beta\times\exp(\mathrm{i}/2\beta\phi)\lambda\right] = 0 \tag{16a}$$

$$\partial_+ V_2 - \theta(x_1)\left[-\mathrm{i}\beta\partial_+\phi V_2 + \sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda V_1 + 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)V_2^2\right] = 0 \tag{16b}$$

$$\partial_- V_1 + \theta(x_1)\left[\sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}V_2 + \frac{\beta}{2\mathrm{i}}\partial_-\phi V_1 + \sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}V_1^2\right] = 0 \tag{16c}$$

$$\partial_- V_2 + \theta(x_1)\left[-\mathrm{i}\beta\partial_-\phi V_2 + \sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}V_1V_2 - 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\right] = 0 \tag{16d}$$

$$\begin{aligned}&\partial_+ U + \theta(x_1)\left[2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)V_2 + \frac{\beta}{2\mathrm{i}}\partial_+\phi\right] = 0\\ &\partial_- U - \theta(x_1)\left[\sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}V_1 + \frac{\beta}{2\mathrm{i}}\partial_-\phi\right] = 0\end{aligned} \tag{17}$$

for the region $x_1 \neq 0$.

As a consequence of the two conditions valid away from the boundary, using $\partial_+\partial_- U = \partial_-\partial_+ U$ and similarly $\partial_+\partial_-\tilde{U} = \partial_-\partial_+\tilde{U}$, we get the conservation equations valid at a general point x:

$$\begin{aligned}&\partial_+\left[\theta(x_1)\left\{\sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}V_1 + \frac{\beta}{2\mathrm{i}}\partial_-\phi\right\}\right]\\ &\qquad = -\partial_-\left[\theta(x_1)\left\{2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)V_2 + \frac{\beta}{2\mathrm{i}}\partial_+\phi\right\}\right]\end{aligned} \tag{18}$$

along with

$$\begin{aligned}&\partial_-\left[\theta(x_1)\left\{\sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}\tilde{V}_1 + \frac{\beta}{2\mathrm{i}}\partial_+\phi\right\}\right]\\ &\qquad = -\partial_+\left[\theta(x_1)\left\{2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\tilde{V}_2 + \frac{\beta}{2\mathrm{i}}\partial_-\phi\right\}\right].\end{aligned} \tag{19}$$

Let us now consider (16*a*) and (16*b*) and substitute

$$V_1 = \sum_{n=1}^{\infty}\lambda^n a_n \qquad V_2 = \sum_{n=1}^{\infty}\lambda^n b_n$$

which yields

$$\partial_+ a_r - \theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_+\phi a_r + 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\sum_{k=1}^{r-1}a_k b_{r-k}\right] = 0 \tag{20}$$

$$\partial_+ b_r - \theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_+\phi b_r + \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)a_{r-1} + 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\sum_{k=1}^{r-1}b_r b_{r-k}\right] = 0. \tag{21}$$

If we assume $x_1 > 0$ then (20) and (21) lead to

$$\begin{aligned}a_1 &= -\sqrt{2\pi}\beta\mathrm{e}^{\frac{\beta}{2\mathrm{i}}\phi}\int_{-\infty}^{x_+}\mathrm{e}^{\mathrm{i}\beta\phi}\,\mathrm{d}x_+\\ b_1 &= \mathrm{e}^{-\mathrm{i}\beta\phi}\\ a_2 &= 2\sqrt{2}\pi\beta^2\mathrm{e}^{-\frac{\mathrm{i}\beta}{2}\phi}\int_{-\infty}^{x_+}\mathrm{e}^{-3\mathrm{i}\beta\phi}\left(\int_{-\infty}^{x'_+}\mathrm{e}^{\mathrm{i}\beta\phi}\,\mathrm{d}x_+\right)\mathrm{d}x'_+\\ b_2 &= 2\pi\beta^2\mathrm{e}^{-\mathrm{i}\beta\phi}\int_{-\infty}^{x_+}\mathrm{e}^{\mathrm{i}\beta\phi}\left(\int_{-\infty}^{x'_+}\mathrm{e}^{\mathrm{i}\beta\phi}\,\mathrm{d}x_+\right)\mathrm{d}x'_+ + 2\pi\beta\mathrm{e}^{-\mathrm{i}\beta\phi}\int_{-\infty}^{x_+}\mathrm{e}^{-2\mathrm{i}\beta\phi}\,\mathrm{d}x_+\end{aligned} \tag{22}$$

and so on. Since the coefficients can be determined explicitly the infinite number of conserved quantities are completely determined in the presence of the boundary condition but away from the point $x_1 = 0$. However, when one wants to investigate the behaviour of these conservation laws at the point $x_1 = 0$, we use equations (8) with a δ function to construct the Riccati equations. In the present situation we get

$$
\begin{aligned}
\partial_- V_1 &= -\theta(x_1)\left[\sqrt{2\pi}\beta \exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}V_2 + \frac{\beta}{2\mathrm{i}}\partial_-\phi V_1 + \sqrt{2\pi}\beta \times \exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}V_1^2\right] \\
&\quad +\delta(x_1)\lambda^{-1}(V_1+V_2)f_2 \\
\partial_- V_2 &= -\theta(x_1)\left[-\beta\partial_-\phi V_2 + \sqrt{2\pi}\beta\exp\left(\frac{\mathrm{i}}{2}\beta\phi\right)\lambda^{-1}V_1V_2 - 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\right] \\
&\quad -\delta(x_1)\lambda^{-1}[f_3 - f_2V_1V_2] \\
\partial_- U &= \theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_-\phi + \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}V_1\right] - \delta(x_1)\lambda^{-1}f_2V_1 \\
\partial_+\tilde{U} &= -\theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_+\phi + \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}V_1\right] - \delta(x_1)\lambda^{-1}f_2V_1
\end{aligned}
\tag{23}
$$

whereas the equations involving ∂_+V_1, ∂_+V_2, $\partial_-\tilde{V}_1$, $\partial_-\tilde{V}_2$, ∂_+U, $\partial_-\tilde{U}$ remain unchanged but

$$
\begin{aligned}
\partial_+\tilde{V}_1 &= -\theta(x_1)\left[\sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}(\tilde{V}_2 + \tilde{V}_1^2) + \frac{\beta}{2\mathrm{i}}\partial_+\phi\tilde{V}_1\right] + \delta(x_1)\lambda^{-1}f_2(\tilde{V}_1+\tilde{V}_2) \\
\partial_+\tilde{V}_2 &= -\theta(x_1)[-\mathrm{i}\beta\partial_+\phi\tilde{V}_2 + \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}\tilde{V}_1\tilde{V}_2 - 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)] \\
&\quad -\delta(x_1)\lambda^{-1}[f_3 - f_2\tilde{V}_1\tilde{V}_2].
\end{aligned}
\tag{24}
$$

As a consequence of this modified set we deduce the following equations which indicate the change in the form of the conserved quantities. If at $x_1 = 0$, we impose $F\psi = \tilde{F}\tilde{\psi}$ we get

$$
\begin{aligned}
&\partial_-\left\{\theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_+\phi V_1 + 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)V_1V_2 - \sqrt{2\pi}\beta\times\exp(\mathrm{i}/2\beta\phi)\lambda\right]\right\} \\
&\quad +\partial_+\left\{\theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_-\phi\tilde{V}_1 + 2\sqrt{\pi}\beta\times\exp(-\mathrm{i}\beta\phi)\tilde{V}_1\tilde{V}_2 - \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda\right]\right\} \\
&\quad +\partial_+\left\{\theta(x_1)\left[\sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}V_2 + \frac{\beta}{2\mathrm{i}}\partial_-\phi V_1\right.\right. \\
&\quad \left.\left.+\sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}V_1^2\right]\right\} \\
&\quad +\partial_-\left\{\theta(x_1)\left[\sqrt{2\pi}\beta\times\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}\tilde{V}_2 + \frac{\beta}{2\mathrm{i}}\partial_+\phi\tilde{V}_1\right.\right. \\
&\quad \left.\left.+\sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}V_1^2\right]\right\} \\
&\quad -\partial_+[\delta(x_1)\lambda^{-1}f_2(V_1+V_2)] - \partial_-[\delta(x_1)\lambda^{-1}f_2(\tilde{V}_1+\tilde{V}_2)] = 0
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\partial_-\left\{\theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_+\phi + 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)V_2\right]\right\} \\
&\quad +\partial_+\left\{\theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_-\phi + 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\tilde{V}_2\right]\right\} \\
&\quad +\partial_+\left\{\theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_-\phi + \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}V_1\right]\right\} \\
&\quad +\partial_-\left\{\theta(x_1)\left[\frac{\beta}{2\mathrm{i}}\partial_+\phi + \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}\tilde{V}_1\right]\right\} - \partial_+[\delta(x_1)\lambda^{-1}f_2V_1]
\end{aligned}
$$

$$-\partial_-[\delta(x_1)\lambda^{-1} f_2 \tilde{V}_1] = 0. \tag{26}$$

And also

$$\begin{aligned}
&\partial_- \left\{\theta(x_1)\left[-\mathrm{i}\beta\partial_+\phi V_2 + \sqrt{2\pi}\beta \exp(\mathrm{i}/2\beta\phi)\lambda V_1 + 2\sqrt{\pi}\exp(-\mathrm{i}\beta\phi)V_2^2\right]\right\} \\
&\quad +\partial_+\left\{\theta(x_1)\left[-\beta\partial_-\phi\tilde{V}_2 + \sqrt{2\pi}\beta\exp\left(-\frac{\mathrm{i}}{2}\beta\phi\right)\lambda\tilde{V}_1\right.\right. \\
&\quad \left.\left. +2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\tilde{V}_2^2\right]\right\} \\
&\quad +\partial_+\left\{\theta(x_1)\left[-\beta\mathrm{i}\partial_+\phi V_2 + \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\times\lambda^{-1}V_1V_2\right.\right. \\
&\quad \left.\left. -2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\right]\right\} \\
&\quad +\partial_-\left\{\theta(x_1)\left[-\beta\partial_+\phi\tilde{V}_2 + \sqrt{2\pi}\beta\exp(\mathrm{i}/2\beta\phi)\lambda^{-1}\tilde{V}_1\tilde{V}_2 - 2\sqrt{\pi}\beta\exp(-\mathrm{i}\beta\phi)\right]\right\} \\
&\quad +\partial_+[\delta(x_1)\lambda^{-1}(f_3 - f_2V_1V_2)] + \partial_-\{\delta(x_1)\lambda^{-1}[f_3 - f_2\tilde{V}_1\tilde{V}_2]\} = 0
\end{aligned} \tag{27}$$

which are obviously well defined on the whole line. Note that the forms of the conservation laws given by equations (25)–(27) are very similar to the original one but modified by the presence of boundary terms. If we now substitute the expansions for V_1, V_2, $\tilde{V}_1$, $\tilde{V}_2$ as before then the coefficients can again be determined *ab initio*. A similar computation can convince one that it effectively means that the original a_n are replaced by $a'_n\theta(x_1) + b'_n\delta(x_1)$. So, in general, one can conclude that the conserved current contains two parts; one valid for general values of x and the other at the point $x_1 = 0$. Written explicitly it reads

$$J = J_1(x)\theta(x_1) + J'_1(x)\delta(x_1).$$

So from our analysis it is possible to reach a conclusion about the integrability of the Dodd–Ballough equation in the Lax sense, even in the presence of the boundary condition—this condition itself being determined from the consistency condition. Additional information is gained about the structure of the conserved quantities which could not be deduced from the approach adopted by Sklyanin.

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